

A Constructive Characterization of Solvable Polynomial Algebras *

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Abstract. For the solvable polynomial algebras introduced and studied by Kandri-Rody and Weispfenning [J. Symbolic Comput., 9(1990)], a constructive characterization is given in terms of Gröbner bases for ideals of free algebras, thereby solvable polynomial algebras are completely determinable in a computational way.

MSC 2010 Primary 16Z05; Secondary 68W30.

Key words PBW basis, Monomial ordering, Gröbner basis, Solvable polynomial algebra.

1. Introduction

In the late 1980s, the celebrated Gröbner basis theory developed by Bruno Buchberger [Bu1, 2] for commutative polynomial ideals was successfully generalized to one-sided ideals in enveloping algebras of Lie algebras by Apel and Lassner [AL], to one-sided ideals in Weyl algebras (including algebras of partial differential operators with polynomial coefficients over a field of characteristic 0) by Galligo [Gal], and more generally, to one-sided and two-sided ideals in solvable polynomial algebras (or algebras of solvable type) by Kandri-Rody and Weispfenning [K-RW]. The class of solvable polynomial algebras includes not only the commutative polynomial algebras, the Weyl algebras, the enveloping algebras of finite dimensional Lie algebras, and a large number of iterated Ore extensions, but also numerous other significant noncommutative algebras (cf. [Li1], [BGV], [Li4]).

Let K be a field and $R = K[X_1, \dots, X_n]$ the commutative polynomial K -algebra in n variables. Originally, a noncommutative solvable polynomial algebra R' was defined in [K-RW] by first fixing a monomial ordering \prec on the standard K -basis $\mathcal{B} = \{X_1^{\alpha_1} \cdots X_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$ of R ,

*Project supported by the National Natural Science Foundation of China (10971044).

and then introducing a new multiplication $*$ on R , such that certain axioms ([K-RW], AXIOMS 1.2) are satisfied. In the formal language of associative K -algebras, a solvable polynomial algebra can actually be defined as a finitely generated associative K -algebra $A = K[a_1, \dots, a_n]$, that has the PBW K -basis $\mathcal{B} = \{a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$ and a (two-sided) monomial ordering \prec on \mathcal{B} such that for $1 \leq i < j \leq n$, $a_j a_i = \lambda_{ji} a_i a_j + f_{ji}$ and $\mathbf{LM}(f_{ji}) \prec a_i a_j$, where $\lambda_{ji} \in K - \{0\}$ and $f_{ji} \in K\text{-span}\mathcal{B}$ ([LW], Definition 2.1). Full details on this definition will be recalled in the next section.

Let $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$ be the noncommutative free K -algebra on $X = \{X_1, \dots, X_n\}$, and $\mathbb{B} = \{1, X_{i_1} X_{i_2} \cdots X_{i_s} \mid X_{i_j} \in X, s \geq 1\}$ the standard K -basis of $K\langle X \rangle$. For convenience, we use capital letters U, V, W, S, \dots to denote elements (monomials) of \mathbb{B} . Recall that a monomial ordering \prec' on \mathbb{B} is a well-ordering such that for $W, U, V \in \mathbb{B}$, $U \prec' V$ implies $WU \prec' WV$, $UW \prec' VW$; and moreover, if $W, U, V, S \in \mathbb{B}$ with $W \neq V$, then $W = UVS$ implies $V \prec' W$ (thereby $1 \prec' W$ for all $1 \neq W \in \mathbb{B}$). As before, we let $R = K[X_1, \dots, X_n]$ denote the commutative polynomial algebra in variables X_1, \dots, X_n , and write $\mathcal{B} = \{X_1^{\alpha_1} \cdots X_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$ for the standard K -basis of R . In [K-RW] it was shown how to construct a solvable polynomial algebra by means of a fixed monomial ordering \prec on \mathcal{B} and a *commutation system* $Q = \{q_{ij} \mid 1 \leq i < j \leq n\} \subset K\langle X \rangle$, where each $q_{ij} = X_j X_i - c_{ij} X_i X_j - p_{ij}$ with $c_{ij} \in K - \{0\}$ and $p_{ij} \in R$ such that $p_{ij} \prec X_i X_j$, that is,

- ([K-RW], Theorem 1.7) if $I(Q)$ denotes the (two-sided) ideal of $K\langle X \rangle$ generated by Q , then the quotient algebra $K\langle X \rangle / I(Q)$ is isomorphic to a solvable polynomial algebra R' defined with respect to the given \prec on \mathcal{B} and the new multiplication $*$ on R satisfying ([K-RW], AXIOMS 1.2) if and only if $I(Q)$ satisfies the condition

$$(H) \quad I(Q) \text{ contains no nonzero commutative polynomials.}$$

With a fixed monomial ordering \prec on \mathcal{B} and a given commutation system Q as above, the relation between the condition (H) and Gröbner bases of $I(Q)$ in $K\langle X \rangle$ was also explored in [K-RW], that is,

- ([K-RW], Theorem 1.11 due to Mora) assuming that there is a positive monomial ordering \prec' on \mathbb{B} which extends the given monomial ordering \prec on \mathcal{B} , such that

$$X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_j^{\alpha_j} \prec' X_j X_i \text{ for all monomials } X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_j^{\alpha_j}, 1 \leq i < j \leq n,$$

then $I(Q)$ satisfies the condition (H) if and only if Q is a Gröbner basis for $I(Q)$ with respect to the monomial ordering \prec' on \mathbb{B} .

As an example, in the case that \prec is the pure lexicographic ordering on \mathcal{B} , the existence of such a positive monomial ordering \prec' on \mathbb{B} was given by Mora (see the ordering \prec' described in [K-RW] after Corollary 1.12).

In the case that $Q = \{X_j X_i - X_i X_j - p_{ij}(X_1, \dots, X_{j-1}) \mid 1 \leq i < j \leq n\} \subset K\langle X \rangle$ but with each $p_{ij}(X_1, \dots, X_{j-1})$ being an element of the free K -algebra $K\langle X_1, \dots, X_{j-1} \rangle$ on $\{X_1, \dots, X_{j-1}\}$, it was proved that

- ([K-RW], Theorem 1.13) the quotient algebra $K\langle X \rangle / I(Q)$ is a solvable polynomial algebra of strictly lexicographical type if and only if $I(Q)$ satisfies the condition (H).

In [Li1, 4], some results on the construction of solvable polynomial algebras by means of Gröbner bases in $K\langle X \rangle$ were also given ([Li1], CH.III, Proposition 2.2, Proposition 2.3; [Li4], Ch.4, Proposition 4.2). However, so far there seems no a complete constructive characterization of solvable polynomial algebras in terms of Gröbner bases for ideals of free algebras, by which one may effectively determine all such algebras. In this note, we solve this problem in Section 2 by Theorem 2.5. Furthermore, we remark in Section 3 that the main results obtained for quadric solvable polynomial algebras in [Li1] indeed hold true for arbitrary solvable polynomial algebras; and we remark that the algebras satisfying the equivalent conditions of Proposition 2.4, especially the algebras described in Theorem 2.5(ii)(a), also provide us with an interesting class of algebras in the computational noncommutative algebra.

Throughout this paper, K denotes a field, $K^* = K - \{0\}$; \mathbb{N} denotes the set of all nonnegative integers. Moreover, the Gröbner basis theory for ideals of free algebras is referred to [Mor].

2. The Main Result

We first briefly recall from ([K-RW], [LW], [Li1]) some basics concerning solvable polynomial algebras. Let $A = K[a_1, \dots, a_n]$ be a finitely generated K -algebra with the minimal set of generators $\{a_1, \dots, a_n\}$. If, for some permutation $\tau = i_1 i_2 \dots i_n$ of $1, 2, \dots, n$, the set $\mathcal{B} = \{a^\alpha = a_{i_1}^{\alpha_1} \dots a_{i_n}^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$, forms a K -basis of A , then \mathcal{B} is referred to as a *PBW K -basis* of A . It is clear that if A has a PBW K -basis, then we can always assume that $i_1 = 1, \dots, i_n = n$. Thus, we make the following convention once for all.

Convention From now on in this paper, if we say that the algebra A has the PBW K -basis \mathcal{B} , then \mathcal{B} is meant the one

$$\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \dots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, adopting the commonly used terminology in computational algebra, elements of \mathcal{B} are referred to as *monomials* of A .

Suppose that A has the PBW K -basis \mathcal{B} as presented above and that \prec is a total ordering on \mathcal{B} . Then every nonzero element $f \in A$ has a unique expression

$$f = \lambda_1 a^{\alpha(1)} + \lambda_2 a^{\alpha(2)} + \dots + \lambda_m a^{\alpha(m)}, \quad \lambda_j \in K^*, \quad a^{\alpha(j)} = a_1^{\alpha_{1j}} a_2^{\alpha_{2j}} \dots a_n^{\alpha_{nj}} \in \mathcal{B}, \quad 1 \leq j \leq m,$$

in which the basis elements satisfy $a^{\alpha(1)} \prec a^{\alpha(2)} \prec \dots \prec a^{\alpha(m)}$. It follows that we may define

$$\begin{aligned}\mathbf{LM}(f) &= a^{\alpha(m)}, & \text{the leading monomial of } f; \\ \mathbf{LC}(f) &= \lambda_m, & \text{the leading coefficient of } f; \\ \mathbf{LT}(f) &= \lambda_m a^{\alpha(m)}, & \text{the leading term of } f.\end{aligned}$$

2.1. Definition Suppose that the K -algebra $A = K[a_1, \dots, a_n]$ has the PBW K -basis \mathcal{B} . If \prec is a total ordering on \mathcal{B} that satisfies the following three conditions:

- (1) \prec is a well-ordering;
- (2) For $a^\gamma, a^\alpha, a^\beta \in \mathcal{B}$, if $a^\alpha \prec a^\beta$ and $\mathbf{LM}(a^\gamma a^\alpha), \mathbf{LM}(a^\gamma a^\beta) \notin K$, then $\mathbf{LM}(a^\gamma a^\alpha) \prec \mathbf{LM}(a^\gamma a^\beta)$;
- (3) For $a^\gamma, a^\alpha, a^\beta \in \mathcal{B}$, if $a^\beta \neq a^\gamma$, and $a^\gamma = \mathbf{LM}(a^\alpha a^\beta)$, then $a^\beta \prec a^\gamma$ (thereby $1 \prec a^\gamma$ for all $a^\gamma \neq 1$),

then we call \prec a *left monomial ordering* on \mathcal{B} .

Similarly, a *right monomial ordering* on \mathcal{B} may be defined.

If \prec is both a left monomial ordering and a right monomial ordering on \mathcal{B} , then we call \prec a *two-sided monomial ordering* (or simply a *monomial ordering*) on \mathcal{B} .

Remark Examples of left (right) monomial orderings on PBW K -bases, which are not right (left) monomial orderings, are given in [Li3].

2.2. Definition Suppose that the K -algebra $A = K[a_1, \dots, a_n]$ has the PBW K -basis \mathcal{B} and that \prec is a (two-sided) monomial ordering on \mathcal{B} . If for all $a^\alpha = a_1^{\alpha_1} \dots a_n^{\alpha_n}$, $a^\beta = a_1^{\beta_1} \dots a_n^{\beta_n} \in \mathcal{B}$, the following holds:

$$\begin{aligned}a^\alpha a^\beta &= \lambda_{\alpha, \beta} a^{\alpha + \beta} + f_{\alpha, \beta}, \\ \text{where } \lambda_{\alpha, \beta} &\in K^*, \quad a^{\alpha + \beta} = a_1^{\alpha_1 + \beta_1} \dots a_n^{\alpha_n + \beta_n}, \text{ and either } f_{\alpha, \beta} = 0 \text{ or} \\ f_{\alpha, \beta} &\in K\text{-span}\mathcal{B} \text{ with } \mathbf{LM}(f_{\alpha, \beta}) \prec a^{\alpha + \beta},\end{aligned}$$

then A is said to be a *solvable polynomial algebra*.

The results of the next proposition are summarized from ([K-RW], Sections 2 – 5).

2.3. Proposition Let $A = K[a_1, \dots, a_n]$ be a solvable polynomial algebra with the (two-sided) monomial ordering \prec on the PBW K -basis \mathcal{B} of A . The following statements hold.

- (i) A is a (left and right) Noetherian domain.
- (ii) Every left ideal I of A has a finite left Gröbner basis $\mathcal{G} = \{g_1, \dots, g_t\}$ in the sense that
 - if $0 \neq f \in I$, then there is some $g_i \in \mathcal{G}$ such that $\mathbf{LM}(g_i) | \mathbf{LM}(f)$, i.e., there is some $a^\gamma \in \mathcal{B}$ such that $\mathbf{LM}(f) = \mathbf{LM}(a^\gamma \mathbf{LM}(g_i))$, or equivalently, with $\gamma(i_j) = (\gamma_{i_{1j}}, \gamma_{i_{2j}}, \dots, \gamma_{i_{nj}}) \in \mathbb{N}^n$, f has a left Gröbner representation:

$$\begin{aligned}f &= \sum_{i,j} \lambda_{ij} a^{\gamma(i_j)} g_j, \text{ where } \lambda_{ij} \in K^*, \quad a^{\gamma(i_j)} \in \mathcal{B}, \quad g_j \in \mathcal{G}, \\ &\text{satisfying } \mathbf{LM}(a^{\gamma(i_j)} g_j) \preceq \mathbf{LM}(f) \text{ for all } (i, j).\end{aligned}$$

(iii) The Buchberger's Algorithm, that computes a finite Gröbner basis for a finitely generated commutative polynomial ideal, has a complete noncommutative version that computes a finite left Gröbner basis for a finitely generated left ideal $I = \sum_{i=1}^m Af_i$ of A .

(iv) Similar results of (ii) and (iii) hold for right ideals and two-sided ideals of A .

□

It is clear that a solvable polynomial algebra A depends on two independent factors, namely a PBW K -basis \mathcal{B} and an appropriate monomial ordering \prec on \mathcal{B} . Let $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$ be the free K -algebra on $X = \{X_1, \dots, X_n\}$ and $\mathbb{B} = \{1, X_{i_1} \cdots X_{i_s} \mid X_{i_j} \in X, s \geq 1\}$ the standard K -basis of $K\langle X \rangle$. Let I be an ideal of $K\langle X \rangle$. Concerning the relation between Gröbner bases of I and the existence of a PBW K -basis for the quotient algebra $A = K\langle X \rangle / I$, we recall the following

2.4. Proposition ([Li4], Ch 4, Theorem 3.1) Let $A = K\langle X \rangle / I$ be as above. Suppose that I contains a subset of $\frac{n(n-1)}{2}$ elements

$$G = \{g_{ji} = X_j X_i - F_{ji} \mid F_{ji} \in K\langle X \rangle, 1 \leq i < j \leq n\}$$

such that with respect to some monomial ordering \prec_x on \mathbb{B} , $\mathbf{LM}(g_{ji}) = X_j X_i$ holds for all the g_{ji} . The following two statements are equivalent:

(i) A has the PBW K -basis $\mathcal{B} = \{\overline{X}_1^{\alpha_1} \overline{X}_2^{\alpha_2} \cdots \overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$ where each \overline{X}_i denotes the coset of I represented by X_i in $K\langle X \rangle / I$.

(ii) Any subset \mathcal{G} of I containing G is a Gröbner basis for I with respect to \prec_x . □

Remark Obviously, Proposition 2.4 holds true if we use any permutation $\{X_{k_1}, \dots, X_{k_n}\}$ of $\{X_1, \dots, X_n\}$. So, in what follows we conventionally use only $\{X_1, \dots, X_n\}$.

We note that that if $G = \{g_{ji} = X_j X_i - F_{ji} \mid F_{ji} \in K\langle X \rangle, 1 \leq i < j \leq n\}$ is a Gröbner basis of the ideal I such that $\mathbf{LM}(g_{ji}) = X_j X_i$ for all the g_{ji} , then the *reduced Gröbner basis* of I is of the form

$$\mathcal{G} = \left\{ g_{ji} = X_j X_i - \sum_q \mu_q^{ji} X_1^{\alpha_{1q}} X_2^{\alpha_{2q}} \cdots X_n^{\alpha_{nq}} \mid \mathbf{LM}(g_{ji}) = X_j X_i, 1 \leq i < j \leq n \right\}$$

where $\mu_q^{ji} \in K$ and $(\alpha_{1q}, \alpha_{2q}, \dots, \alpha_{nq}) \in \mathbb{N}^n$. Bearing in mind Definition 2.2 and combining this fact, we have the following characterization of solvable polynomial algebras in terms of Gröbner bases for ideals of free algebras.

2.5. Theorem Let $A = K[a_1, \dots, a_n]$ be a finitely generated algebra over the field K , and let $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$ be the free K -algebras with the standard K -basis $\mathbb{B} = \{1, X_{i_1} \cdots X_{i_s} \mid X_{i_j} \in X, s \geq 1\}$. With notation as before, the following two statements are equivalent:

(i) A is a solvable polynomial algebra in the sense of Definition 2.2.

(ii) $A \cong \overline{A} = K\langle X \rangle / I$ via the K -algebra epimorphism $\pi_1: K\langle X \rangle \rightarrow A$ with $\pi_1(X_i) = a_i$, $1 \leq i \leq n$, $I = \text{Ker}\pi_1$, satisfying

(a) with respect to some monomial ordering \prec_X on \mathbb{B} , the ideal I has a finite Gröbner basis G and the reduced Gröbner basis of I is of the form

$$\mathcal{G} = \left\{ \begin{array}{l} g_{ji} = X_j X_i - \lambda_{ji} X_i X_j - F_{ji} \\ \text{with } F_{ji} = \sum_q \mu_q^{ji} X_1^{\alpha_{1q}} X_2^{\alpha_{2q}} \cdots X_n^{\alpha_{nq}} \end{array} \middle| \begin{array}{l} \mathbf{LM}(g_{ji}) = X_j X_i, \\ 1 \leq i < j \leq n \end{array} \right\}$$

where $\lambda_{ji} \in K^*$, $\mu_q^{ji} \in K$, and $(\alpha_{1q}, \alpha_{2q}, \dots, \alpha_{nq}) \in \mathbb{N}^n$, thereby $\mathcal{B} = \{\overline{X}_1^{\alpha_1} \overline{X}_2^{\alpha_2} \cdots \overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$ forms a PBW K -basis for \overline{A} , where each \overline{X}_i denotes the coset of I represented by X_i in \overline{A} ; and

(b) there is a (two-sided) monomial ordering \prec on \mathcal{B} such that $\mathbf{LM}(\overline{F}_{ji}) \prec \overline{X}_i \overline{X}_j$ whenever $\overline{F}_{ji} \neq 0$, where $\overline{F}_{ji} = \sum_q \mu_q^{ji} \overline{X}_1^{\alpha_{1q}} \overline{X}_2^{\alpha_{2q}} \cdots \overline{X}_n^{\alpha_{nq}}$, $1 \leq i < j \leq n$.

Proof (i) \Rightarrow (ii) Let $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ be the PBW K -basis of the solvable polynomial algebra A and \prec a (two-sided) monomial ordering on \mathcal{B} . By Definition 2.2, the generators of A satisfy the relations:

$$a_j a_i = \lambda_{ji} a_i a_j + f_{ji}, \quad 1 \leq i < j \leq n, \quad (*)$$

where $\lambda_{ji} \in K^*$ and $f_{ji} = \sum_q \mu_q^{ji} a^{\alpha(q)} \in K\text{-span}\mathcal{B}$ with $\mathbf{LM}(f_{ji}) \prec a_i a_j$. Consider in the free K -algebra $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$ the subset

$$\mathcal{G} = \{g_{ji} = X_j X_i - \lambda_{ji} X_i X_j - F_{ji} \mid 1 \leq i < j \leq n\},$$

where if $f_{ji} = \sum_q \mu_q^{ji} a_1^{\alpha_{1q}} a_2^{\alpha_{2q}} \cdots a_n^{\alpha_{nq}}$ then $F_{ji} = \sum_q \mu_q^{ji} X_1^{\alpha_{1q}} X_2^{\alpha_{2q}} \cdots X_n^{\alpha_{nq}}$ for $1 \leq i < j \leq n$. We write $J = \langle \mathcal{G} \rangle$ for the ideal of $K\langle X \rangle$ generated by \mathcal{G} and put $\overline{A} = K\langle X \rangle / J$. Let $\pi_1: K\langle X \rangle \rightarrow A$ be the K -algebra epimorphism with $\pi_1(X_i) = a_i$, $1 \leq i \leq n$, and let $\pi_2: K\langle X \rangle \rightarrow \overline{A}$ be the canonical algebra epimorphism. It follows from the fundamental theorem of homomorphism that there is an algebra epimorphism $\varphi: \overline{A} \rightarrow A$ defined by $\varphi(\overline{X}_i) = a_i$, $1 \leq i \leq n$, such that the following diagram of algebra homomorphisms is commutative:

$$\begin{array}{ccc} K\langle X \rangle & \xrightarrow{\pi_2} & \overline{A} \\ \pi_1 \downarrow & \swarrow \varphi & \varphi \circ \pi_2 = \pi_1 \\ A & & \end{array}$$

On the other hand, by the definition of each g_{ji} we see that every element $\overline{H} \in \overline{A}$ may be written as $\overline{H} = \sum_j \mu_j \overline{X}_1^{\beta_{1j}} \overline{X}_2^{\beta_{2j}} \cdots \overline{X}_n^{\beta_{nj}}$ with $\mu_j \in K$ and $(\beta_{1j}, \dots, \beta_{nj}) \in \mathbb{N}^n$, where each \overline{X}_i is the coset of J represented by X_i in \overline{A} . Noticing the relations presented in (*), it is straightforward to check that the correspondence

$$\begin{array}{ccc} \psi: & A & \longrightarrow \overline{A} \\ & \sum_i \lambda_i a_1^{\alpha_{1i}} \cdots a_n^{\alpha_{ni}} & \mapsto \sum_i \lambda_i \overline{X}_1^{\alpha_{1i}} \cdots \overline{X}_n^{\alpha_{ni}} \end{array}$$

is an algebra homomorphism such that $\varphi \circ \psi = 1_A$ and $\psi \circ \varphi = 1_{\overline{A}}$, where 1_A and $1_{\overline{A}}$ denote the multiplicative identities of A and \overline{A} respectively. This shows that $A \cong \overline{A}$, thereby $\text{Ker}\pi_1 = I = J$; moreover, $\mathcal{B} = \{\overline{X}_1^{\alpha_1} \overline{X}_2^{\alpha_2} \cdots \overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$ forms a PBW K -basis for \overline{A} , and \prec is a (two-sided) monomial ordering on \mathcal{B} .

We next show that \mathcal{G} forms the reduced Gröbner basis for I as described in (a). To this end, we first show that the monomial ordering \prec on \mathcal{B} induces a monomial ordering \prec_x on the standard K -basis \mathbb{B} of $K\langle X \rangle$. For convenience, we use capital letters U, V, W, S, \dots to denote elements (monomials) in \mathbb{B} . We also fix a graded lexicographic ordering \prec_{grlex} on \mathbb{B} (with respect to a fixed positively weighted gradation of $K\langle X \rangle$) such that

$$X_1 \prec_{grlex} X_2 \prec_{grlex} \cdots \prec_{grlex} X_n.$$

Then, for $U, V \in \mathbb{B}$ we define

$$U \prec_x V \text{ if } \begin{cases} \mathbf{LM}(\pi_1(U)) \prec \mathbf{LM}(\pi_1(V)), \\ \text{or} \\ \mathbf{LM}(\pi_1(U)) = \mathbf{LM}(\pi_1(V)) \text{ and } U \prec_{grlex} V. \end{cases}$$

Since A is a domain (Proposition 2.3(i)) and π_1 is an algebra homomorphism with $\pi_1(X_i) = a_i$ for $1 \leq i \leq n$, it follows that $\mathbf{LM}(\pi_1(W)) \neq 0$ for all $W \in \mathbb{B}$. We also note from Definition 2.2 that if $f, g \in A$ are nonzero elements, then $\mathbf{LM}(fg) = \mathbf{LM}(\mathbf{LM}(f)\mathbf{LM}(g))$. Thus, if $U, V, W \in \mathbb{B}$ and $U \prec_x V$ subject to $\mathbf{LM}(\pi_1(U)) \prec \mathbf{LM}(\pi_1(V))$, then

$$\begin{aligned} \mathbf{LM}(\pi_1(WU)) &= \mathbf{LM}(\mathbf{LM}(\pi_1(W))\mathbf{LM}(\pi_1(U))) \\ &\prec \mathbf{LM}(\mathbf{LM}(\pi_1(W))\mathbf{LM}(\pi_1(V))) \\ &= \mathbf{LM}(\pi_1(WV)) \end{aligned}$$

implies $WU \prec_x WV$; if $U \prec_x V$ subject to $\mathbf{LM}(\pi_1(U)) = \mathbf{LM}(\pi_1(V))$ and $U \prec_{grlex} V$, then

$$\begin{aligned} \mathbf{LM}(\pi_1(WU)) &= \mathbf{LM}(\mathbf{LM}(\pi_1(W))\mathbf{LM}(\pi_1(U))) \\ &= \mathbf{LM}(\mathbf{LM}(\pi_1(W))\mathbf{LM}(\pi_1(V))) \\ &= \mathbf{LM}(\pi_1(WV)) \end{aligned}$$

and $WU \prec_{grlex} WV$ implies $WU \prec_x WV$. Similarly, if $U \prec_x V$ then $US \prec_x VS$ for all $S \in \mathbb{B}$. Moreover, if $W, U, V, S \in \mathbb{B}$, $W \neq V$, such that $W = UVS$, then $\mathbf{LM}(\pi_1(W)) = \mathbf{LM}(\pi_1(UVS))$ and clearly $V \prec_{grlex} W$, thereby $V \prec_x W$. Since \prec is a well-ordering on \mathcal{B} and \prec_{grlex} is a well-ordering on \mathbb{B} , the above argument shows that \prec_x is a (two-sided) monomial ordering on \mathbb{B} . With this monomial ordering \prec_x in hand, by the definition of F_{ji} we see that $\mathbf{LM}(F_{ji}) \prec_x X_i X_j$. Furthermore, since $\mathbf{LM}(\pi_1(X_j X_i)) = a_j a_i = \mathbf{LM}(\pi_1(X_i X_j))$ and $X_i X_j \prec_{grlex} X_j X_i$, we see that $X_i X_j \prec_x X_j X_i$. It follows that $\mathbf{LM}(g_{ji}) = X_j X_i$ for $1 \leq i < j \leq n$. Now, by Proposition 2.4 we conclude that \mathcal{G} forms a Gröbner basis for I with respect to \prec_x . Finally, by the definition of \mathcal{G} , it is clear that \mathcal{G} is the reduced Gröbner basis of I and \mathcal{G} meets the requirement of Proposition 2.4, hence $\mathcal{B} = \{\overline{X}_1^{\alpha_1} \overline{X}_2^{\alpha_2} \cdots \overline{X}_n^{\alpha_n} \mid \alpha_j \in \mathbb{N}\}$ forms a PBW K -basis for \overline{A} , as desired.

(ii) \Rightarrow (i) Note that (a) + (b) tells us that the generators of \overline{A} satisfy the relations $\overline{X}_j \overline{X}_i = \lambda_{ji} \overline{X}_i \overline{X}_j + \overline{F}_{ji}$, $1 \leq i < j \leq n$, and that if $\overline{F}_{ji} \neq 0$ then $\mathbf{LM}(\overline{F}_{ji}) \prec \overline{X}_i \overline{X}_j$ with respect to the given monomial ordering \prec on \mathcal{B} . It follows that \overline{A} and hence A is a solvable polynomial algebra in the sense of Definition 2.2. \square

Remark The monomial ordering \prec_x we defined in the proof of Theorem 2.5 is a modification of the *lexicographic extension* defined in [EPS]. But our definition of \prec_x involves a graded monomial ordering \prec_{grlex} on the standard K -basis \mathbb{B} of the free K -algebra $K\langle X \rangle = K\langle X_1, \dots, X_n \rangle$. The reason is that *the monomial ordering \prec_x on \mathbb{B} must be compatible with the usual rule of division*, namely, $W, U, V, S \in \mathbb{B}$, $W \neq V$, and $W = UVS$ implies $V \prec_x W$. While it is clear that if we use any lexicographic ordering \prec_{lex} in the definition of \prec_x , then this rule will not work in general.

Note that our discussion on solvable polynomial algebras made use of Definition 2.1 for a monomial ordering and Definition 2.2 for a solvable polynomial algebra. In light of ([K-RW], AXIOMS 1.2), the next proposition may make the practical use of Theorem 2.5 much easier and flexible in determining solvable polynomial algebras.

2.6. Proposition Let $A = K[a_1, \dots, a_n]$ be a finitely K -algebra with the PBW K -basis $\mathcal{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ and the generators of A satisfy the relations:

$$a_j a_i = \lambda_{ji} a_i a_j + f_{ji}, \quad 1 \leq i < j \leq n,$$

where $\lambda_{ji} \in K^*$ and $f_{ji} \in K\text{-span}\mathcal{B}$. Then the following two statements are equivalent.

(i) There is a (two-sided) monomial ordering \prec on \mathcal{B} in the sense of Definition 2.1, such that

$$\mathbf{LM}(f_{ji}) \prec a_i a_j, \quad 1 \leq i < j \leq n.$$

(ii) There is a monomial ordering \prec on the additive monoid \mathbb{N}^n , i.e., \prec is a well-ordering and $\alpha \prec \beta$ implies $\alpha + \gamma \prec \beta + \gamma$ for all $\alpha, \beta, \gamma \in \mathbb{N}^n$, such that if $\mathbf{LM}(f_{ji}) = a^\alpha$ and we write $a^\beta = a_i a_j$, then $\alpha \prec \beta$, $1 \leq i < j \leq n$. \square

We end this section by an example illustrating Theorem 2.5 and Proposition 2.6.

Example 1. Considering the \mathbb{N} -graded structure of the free K -algebra $K\langle X \rangle = K\langle X_1, X_2, X_3 \rangle$ by assigning X_1 the degree 2, X_2 the degree 1 and X_3 the degree 4, let I be the ideal of $K\langle X \rangle$ generated by the elements

$$\begin{aligned} g_1 &= X_1 X_2 - X_2 X_1, \\ g_2 &= X_3 X_1 - \lambda X_1 X_3 - \mu X_3 X_2^2 - f(X_2), \\ g_3 &= X_3 X_2 - X_2 X_3, \end{aligned}$$

where $\lambda \in K^*$, $\mu \in K$, $f(X_2)$ is a polynomial in X_2 which has degree ≤ 6 , or $f(X_2) = 0$. The following properties hold.

- (1) If we use the graded lexicographic ordering $X_2 \prec_{grlex} X_1 \prec_{grlex} X_3$ on $K\langle X \rangle$, then the three generators have the leading monomial $\mathbf{LM}(g_1) = X_1X_2$, $\mathbf{LM}(g_2) = X_3X_1$, and $\mathbf{LM}(g_3) = X_3X_2$. It is straightforward to verify that $\mathcal{G} = \{g_1, g_2, g_3\}$ forms a Gröbner basis for I .
- (2) With respect to the fixed \prec_{grlex} in (1), the reduced Gröbner basis \mathcal{G}' of I consists of

$$\begin{aligned} g_1 &= X_1X_2 - X_2X_1, \\ g_2 &= X_3X_1 - \lambda X_1X_3 - \mu X_2^2X_3 - f(X_2), \\ g_3 &= X_3X_2 - X_2X_3, \end{aligned}$$

(3) Writing $A = K[a_1, a_2, a_3]$ for the quotient algebra $K\langle X \rangle/I$, where a_1 , a_2 and a_3 denote the cosets $X_1 + I$, $X_2 + I$ and $X_3 + I$ in $K\langle X \rangle/I$ respectively, it follows that A has the PBW basis $\mathcal{B} = \{a^\alpha = a_2^{\alpha_2}a_1^{\alpha_1}a_3^{\alpha_3} \mid \alpha = (\alpha_2, \alpha_1, \alpha_3) \in \mathbb{N}^3\}$. Noticing that $a_2a_1 = a_1a_2$, it is clear that $\mathcal{B}' = \{a^\alpha = a_1^{\alpha_1}a_2^{\alpha_2}a_3^{\alpha_3} \mid \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3\}$ is also a PBW basis for A . Since $a_3a_1 = \lambda a_1a_3 + \mu a_2^2a_3 + f(a_2)$, where $f(a_2) \in K\text{-span}\{1, a_2, a_2^2, \dots, a_2^6\}$, we see that A has the monomial ordering \prec_{lex} on \mathcal{B}' , which is given by the lexicographic ordering \prec_{lex} on \mathbb{N}^3 such that $a_3 \prec_{lex} a_2 \prec_{lex} a_1$ and $\mathbf{LM}(\mu a_2^2a_3 + f(a_2)) \prec_{lex} a_1a_3$, thereby A is turned into a solvable polynomial algebra with respect to \prec_{lex} .

Moreover, one easily checks that if a_1 is assigned the degree 2, a_2 is assigned the degree 1 and a_3 is assigned the degree 4, then, A has another monomial ordering \prec_{grlex} on \mathcal{B}' , which is given by the graded lexicographic ordering \prec_{grlex} on \mathbb{N}^3 such that $a_3 \prec_{grlex} a_2 \prec_{grlex} a_1$ and $\mathbf{LM}(\mu a_2^2a_3 + f(a_2)) \prec_{grlex} a_1a_3$, thereby A is turned into a solvable polynomial algebra with respect to \prec_{grlex} .

3. Further Remarks and Questions

First recall from [Li1] that a *quadric solvable polynomial algebra* is a solvable polynomial algebra $A = K[a_1, \dots, a_n]$ with the PBW K -basis \mathcal{B} and a *graded monomial ordering* \prec_{gr} subject to the convention that *each a_i is assigned the degree 1*, such that

$$a_ja_i = \lambda_{ji}a_ia_j + \sum_{k \leq \ell} \lambda_{ji}^{k\ell} a_ka_\ell + \sum_h \lambda_h a_h + c_{ji}, \quad 1 \leq i < j \leq n,$$

where $\lambda_{ji} \in K^*$, $\lambda_{ji}^{k\ell}, \lambda_h, c_{ji} \in K$; in the case where $\sum_{k \leq \ell} \lambda_{ji}^{k\ell} a_ka_\ell + \sum_h \lambda_h a_h + c_{ji} = 0$ for $1 \leq i < j \leq n$, a quadric solvable polynomial algebra A is called a *homogeneous solvable polynomial algebra*. For a quadric solvable polynomial algebra A , by introducing the \prec_{gr} -filtration $\mathcal{F}A$ of A with respect to \prec_{gr} , and passing to the associated graded algebra $G^{\mathcal{F}}(A)$ which is a homogeneous solvable polynomial algebra, ([Li1], CH.V, CH.VI) shows how to calculate the Gelfand-Kirillov dimension $\text{GK.dim} A/L$ and the multiplicity $e(A/L)$ of a cyclic A -module A/L , as well as how to calculate $\text{GK.dim}(A/L \otimes_K A/J)$ and $e(A/L \otimes_K A/J)$, where L and J denote left ideals of A ; moreover, an “elimination lemma for quadric solvable polynomial algebras” is obtained. By standard module theory, the obtained results certainly apply to finitely generated A -modules.

3.1. Remark Let A be an *arbitrary* solvable polynomial algebra A with a monomial ordering \prec (which is unnecessarily a graded monomial ordering). Note that

- (1) the calculation of Gelfand-Kirillov dimension and multiplicity of a module A/L deals only with the \mathbb{N} -filtration of A/L induced by the standard \mathbb{N} -filtration of the K -vector space A ;
- (2) with respect to *any* monomial ordering \prec on A , A has an \prec -filtration $\mathcal{F}A$ that turns A into a \mathcal{B} -filtered ring such that the associated graded algebra $G^{\mathcal{F}}(A)$ is a homogeneous solvable polynomial algebra; and
- (3) the tensor product $A \otimes_K B$ of A with any solvable polynomial algebras B is a solvable polynomial algebra.

If one checks the text of ([Li1], CH.V, CH.VI) carefully, it is not difficult to see that the same (key) result of ([Li], CH.V, Proposition 7.2) holds for A , thereby the main results of ([Li1], CH.V, CH.VI) all holds true for A .

3.2. Remark Furthermore, it was shown in ([Li1, LNM], CH.VIII, Theorem 3.7, Theorem 4.1) that if A is a quadric solvable polynomial algebra with a graded monomial ordering \prec_{gr} , then A has global homological dimension $\text{gl.dim} A \leq n$ and, by the K_0 -part of Quillen's theorem ([Qu], Theorem 7), the K_0 -group of A is isomorphic to the additive group of integers \mathbb{Z} , i.e., $K_0(A) \cong \mathbb{Z}$. Now, it follows from ([Li4], Ch.5, Corollary 7.6), the K_0 -part of Quillen's theorem ([Qu], Theorem 7), and our main result Theorem 2.5 that the same results hold true in a more extended context, that is, we have

3.3. Proposition Let $A = K\langle X \rangle / I$ be an algebra that satisfies the equivalent conditions of Proposition 2.4. Then $\text{gl.dim} A \leq n$, and $K_0(A) \cong \mathbb{Z}$. Consequently, if $A = K[a_1, \dots, a_n]$ is an arbitrary solvable polynomial algebra with a monomial ordering \prec (which is unnecessarily a graded monomial ordering). Then $\text{gl.dim} A \leq n$, and $K_0(A) \cong \mathbb{Z}$.

Before giving a few observations on the algebras that satisfy the equivalent conditions of Proposition 2.4, we write $\mathcal{A}_{\text{gpbw}}$ to denote the class of such algebras for convenience.

Observation (1) If $A = K\langle X_1, \dots, X_n \rangle / I$, $B = K\langle Y_1, \dots, Y_m \rangle / J \in \mathcal{A}_{\text{gpbw}}$, then, by using an appropriate elimination monomial ordering, a similar argument as in the proof (i) \Rightarrow (ii) of Theorem 2.5 turns out that $A \otimes_K B \in \mathcal{A}_{\text{gpbw}}$.

(2) If $A = K\langle X \rangle / I \in \mathcal{A}_{\text{gpbw}}$, the Gröbner basis $\mathcal{G} = \{g_{ji} \mid 1 \leq i < j \leq n\}$ of I is obtained with respect to some graded monomial ordering \prec_{gr} , and in \mathcal{G} each $g_{ji} = X_j X_i - \sum \mu_{k\ell} X_k X_\ell + \sum b_q X_q + c_{ji}$ with $\mu_{k\ell}, b_q, c_{ji} \in K$, and $X_k X_\ell \neq X_j X_i$, then, it follows from ([Li4], Ch. 6, Theorem 3.1) that A is a non-homogeneous Koszul algebra in the sense of [Pr] provided $b_q \in K^*$ (if $b_q = 0, c_{ji} = 0$, then it is well-known that A is a homogeneous Koszul algebra). Consequently, every quadric solvable polynomial algebra A in the sense of [Li1] is either a homogeneous Koszul algebra or a non-homogeneous Koszul algebra (see also [Li2], Example 4.1).

(3) Let A be as in (2) above. By ([Li4], Ch.7, Theorem 3.8), the Rees algebra \tilde{A} of A defined with respect to the standard \mathbb{N} -filtration of A is a homogeneous Koszul algebra.

Thus, from both a computational and a structural viewpoint, our discussion made so far has shown that it is worthwhile to pay more attention to algebras in the class of algebras $\mathcal{A}_{\text{gpbw}}$. Especially for an algebra $A = K\langle X \rangle / I$ as described in Theorem 2.5(ii)(a), we have the following

Questions (1) Is A a domain?

(2) Is A a Noetherian ring?

(3) Is it always possible to define a monomial ordering \prec on the PBW K -basis \mathcal{B} of A via Proposition 2.6, such that A is turned into a solvable polynomial algebra?

References

- [AL] J. Apel and W. Lassner, An extension of Buchberger’s algorithm and calculations in enveloping fields of Lie algebras, *J. Symbolic Comput.*, 6(1988), 361–370.
- [Bu1] B. Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen polynomideal*, PhD thesis, University of Innsbruck, 1965.
- [Bu2] B. Buchberger, Gröbner bases: An algorithmic method in polynomial ideal theory. In: *Multidimensional Systems Theory* (Bose, N.K., ed.), Reidel Dordrecht, 1985, 184–232.
- [BGV] J. Bueso, J. Gómez–Torrecillas, and A. Verschoren, *Algorithmic methods in non-commutative algebra: Applications to quantum groups*. Kluwer Academic Publishers, 2003.
- [EPS] D. Eisenbud, I. Peeva and B. Sturmfels, Non-commutative Gröbner bases for commutative algebras, *Proc. Amer. Math. Soc.*, 126(1998), 687–691.
- [Gal] A. Galligo, Some algorithmic questions on ideals of differential operators, *Proc. EURO-CAL’85*, LNCS 204, 1985, 413–421.
- [K-RW] A. Kandri-Rody and V. Weispfenning, Non-commutative Gröbner bases in algebras of solvable type, *J. Symbolic Comput.*, 9(1990), 1–26.
- [Li1] H. Li, *Noncommutative Gröbner Bases and Filtered-Graded Transfer*, LNM, 1795, Springer-Verlag, Berlin, 2002.
- [Li2] H. Li, The general PBW property, *Alg. Colloquium*, 14(4)(2007), 541–554.
- [Li3] H. Li, Looking for Gröbner basis theory for (almost) skew 2-nomial algebras, *J. Symbolic Computation*, 45(2010), 918–942.
- [Li4] H. Li, *Gröbner Bases in Ring Theory*, World Scientific, 2011.
- [LW] H. Li and Y. Wu, Filtered-graded transfer of Gröbner basis computation in solvable polynomial algebras, *Comm. Alg.*, 28(1)(2000), 15–32.
- [Mor] T. Mora, An introduction to commutative and noncommutative Gröbner Bases, *Theoretic Computer Science*, 134(1994), 131–173.
- [Pr] S. Priddy, Koszul resolutions, *Trans. Amer. Math. Soc.*, 152(1970), 39–60.

- [Qu] D. Quillen, Higher algebraic K -theory I, in *Algebraic K-theory I: Higher K-theory*, ed., H. Bass, Lecture Notes in Mathematics 341, Springer-Verlag, New York-Berlin, 1973, 85–147.